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Modular Invariance in Superstring on Calabi-Yau n-fold with $A-D-E$ Singularity

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Abstract

We study the type II superstring theory on the background $\mathbf{R}^{d-1,1} \times X_n$, where X_n is a Calabi-Yau n-fold ($2n + d = 10$) with an isolated singularity, by making use of the holographically dual description proposed by Giveon-Kutasov-Pelc [1]. We compute the toroidal partition functions for each of the cases $d = 6, 4, 2$, and obtain manifestly modular invariant solutions classified by the standard $A-D-E$ series corresponding to the type of singularities on X_n. Partition functions of these modular invariants all vanish due to theta function identities and are consistent with the presence of space-time supersymmetry.

1 Introduction

String theory on singular backgrounds has been recently receiving much attentions from various view points [1]-[9]. An important feature of a string propagating near singularities is the appearance of light solitons originating from the branes wrapped around some vanishing cycles. This is a typical non-perturbative effect in string theory which is difficult to be worked out from the world-sheet picture of perturbative string theory, even when a decoupling limit $g_s(\equiv e^{\phi(\infty)}) \rightarrow 0$ is taken. In fact, no matter how small g_s is, the VEV of dilaton will blow up at the location of singularity. As was first pointed out in [10], the *vanishing* world-sheet theta angle ($\theta_{ws} \approx \int B = 0$) is essential in the appearance of such a non-perturbative effect. Several authors demonstrated [11, 3, 12, 7] that the conformal theory on string world-sheet becomes singular in this situation. On the other hand, the ordinary "smooth" conformal theory (orbifold CFT [13], $\mathcal{N} = 2$ Landau-Ginzburg model [14, 15] etc.) corresponds to the backgrounds with a *non-vanishing* θ_{ws}. In the latter case the brane wrapped around a collapsed cycle becomes a "fractional brane" [16] with a finite mass, and hence a perturbative approach to string theory is reliable, at least if the string coupling g_s is sufficiently small and the mass of wrapped brane is large.

The first approach to such a singular CFT with a vanishing θ_{ws} was given in refs. [2, 3] which were inspired by the theory of two dimensional black-hole. In these papers it is pointed out that such a singular conformal theory can be described by the Landau-Ginzburg (LG) model with a superpotential including a *negative* power of some chiral superfield, and the subtlety in handling the negative power may be avoided by reformulating it as a Kazama-Suzuki model [17] for the non-compact coset $SL(2;\mathbf{R})/U(1)$.

More recently, a refinement of this approach was given in [1, 4, 5, 6], which is based on a holographic point of view analogous to the AdS/CFT correspondence [18]. In these papers the sector of LG theory with a negative power superpotential is replaced by a suitable linear dilaton background (the $\mathcal{N} = 2$ Liouville theory [19, 20]) describing the throat structure near the singularity, and it is pointed out that the decoupled non-gravitational theory (on the space-time transverse to the singular Calabi-Yau n-fold) has a dual description by a non-critical string theory including the dynamics of Liouville field [19, 20]. This duality is regarded as "holographic" in a manner similar to AdS/CFT in the sense that the throat

variable (Liouville field) ϕ corresponds to an extra non-compact dimension, and the decoupled theory is naturally defined in the weakly coupled region $\phi \sim +\infty$ ("boundary").

This approach to the singular Calabi-Yau compactification is interesting in the sense that the non-critical superstring theory is playing a novel role. The main purpose of this paper is to provide the basic consistency check for these theories: the check of the modular invariance of the toroidal partition function. In the case of $d = 6$ (singular K3 surface) the result is straightforward. The essential part of this case is already studied in [3], and another approach from the standpoint of brane probe is given in [21]. We have the standard $A-D-E$ classification of modular invariants corresponding to the type of degeneration of K3 surface, which coincides exactly with the well-known modular invariants of $SU(2)$ WZW model [22]. This result is not surprising, since the T-duality leads us to the theory of NS5-branes [3, 23] and it is well-known [24] that the world-sheet CFT of string propagation in the background of NS5-branes includes the $SU(2)$ WZW model.

The cases of $d = 4$ and 2 are more difficult to analyse and are the central subjects of this paper. Although we do not have a simple world-sheet interpretation like the $d = 6$ case, we can construct and classify the modular invariants of these string theories. We will find out that the conformal blocks in our models have modular transformation properties analogous to those of parafermion theories (coset CFT of $SU(2)/U(1)$) [25] and we will again obtain the $A - D - E$ classification of modular invariants corresponding to the type of singularities on CY_n. It turns out that the partition functions of our modular invariants all vanish due to some theta function identities, which is consistent with the existence of space-time SUSY.

2 Theory of Singular CY_n-Compactification as Non-critical Superstring

Let us consider type II string theory on the background $\mathbf{R}^{d-1,1} \times X_n$, where X_n is a CY n-fold ($2n + d = 10$) with an isolated singularity (locally) defined by $F(x^1, x^2, \ldots, x^{n+1}) = 0$. As is pointed out in [8, 1], in the decoupling limit $g_s \to 0$ we have a non-gravitational, but non-trivial quantum theory on $\mathbf{R}^{d-1,1}$. This fact contrasts with the cases of a smooth CY_n, where we expect a free theory in the $g_s \to 0$ limit. These d-dimensional quantum theories

are expected to flow to non-trivial conformal fixed points in the IR limit. In the $d = 6$ case, which is essentially the theory of NS5 branes, they become the "little string theory" [26, 5] including non-local excitations. The $d = 4$ case corresponds to the 4-dimensional $\mathcal{N} = 2$ SCFT describing the Argyres-Douglas points on the moduli space of $\mathcal{N} = 2$ SYM_4 [27] (see also [6]). The $d = 2$ case leads to the class of AdS_3 vacua with space-time $\mathcal{N} = 2$ SUSY studied in [28]. Space-time CFT is naturally identified with the boundary CFT in the context of AdS_3/CFT_2 correspondence.

The holographic duality proposed in [1] is represented as follows;

$$\begin{aligned}&\text{decoupling limit of superstring on } \mathbf{R}^{d-1,1} \times X_n \iff \\ &\qquad\qquad\qquad \text{superstring on } \mathbf{R}^{d-1,1} \times (\mathbf{R}_\phi \times S^1) \times LG(W = F),\end{aligned}$$

where $LG(W = F)$ stands for the $\mathcal{N} = 2$ LG model with a superpotential $W = F$. Moreover " $\mathbf{R}_\phi$" indicates a linear dilaton background with the background charge $Q(> 0)$. The throat sector $\mathbf{R}_\phi \times S^1$ is described by the $\mathcal{N} = 2$ Liouville theory [19, 20] whose field contents consist of bosonic variables ϕ (parametrizing $\mathbf{R}_\phi$), Y (parametrizing S^1) and their fermionic partners Ψ^+, Ψ^-. The superconformal currents are written as

$$\left\{\begin{array}{l} T = -\frac{1}{2}(\partial Y)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi - \frac{1}{2}(\Psi^+\partial\Psi^- - \partial\Psi^+\Psi^-) \\ G^\pm = -\frac{1}{\sqrt{2}}\Psi^\pm(i\partial Y \pm \partial\phi) \mp \frac{Q}{\sqrt{2}}\partial\Psi^\pm \\ J = \Psi^+\Psi^- - Qi\partial Y, \end{array}\right. \tag{2.1}$$

which generates the $\mathcal{N} = 2$ superconformal algebra (SCA) with $\hat{c}(\equiv \frac{c}{3}) = 1 + Q^2$.

Since we have a linear dilaton background, $\Phi(\phi) = -\frac{Q}{2}\phi$, the theory is weakly coupled in the "near boundary region" $\phi \sim +\infty$. On the other hand, in the opposite end $\phi \sim -\infty$ (near the singularity) the string coupling blows up, and hence the perturbative approach does not make sense. As is discussed in [1], one must add the "Liouville potential" (or the "cosmological constant term") to the world-sheet action of the Liouville sector,

$$\begin{aligned} S_L \;&\longrightarrow\; S_L + \delta S_+ + \delta S_-, \\ \delta S_\pm &\stackrel{\text{def}}{=} \mu \int d^2z\, \Psi^\pm \bar{\Psi}^\pm e^{-\frac{1}{Q}(\phi \mp iY)}, \end{aligned} \tag{2.2}$$

in order to prevent the string from propagating into the dangerous region $\phi \sim -\infty$. $\delta S_\pm$ are actually the screening charges in the sense that they commute with all the generators of SCA (2.1). So, we shall carry out all the computations as a free conformal theory on the

world-sheet neglecting the existence of Liouville interaction (2.2), although we have to keep in our mind that we cannot set $\mu = 0$[1].

Throughout this paper we focus our attention to cases when X_n has an isolated rational singularity. In these cases the LG theory with $W = F$ is equivalent to the familiar $\mathcal{N} = 2$ minimal models of the $A - D - E$ type corresponding to the classification of rational singularities defined by $F = 0$. These models include the chiral primary fields which are in one-to-one correspondence with the exponents of the $A - D - E$ group, and have the central charge $\hat{c} = \dfrac{N-2}{N}$, where N is the dual Coxeter number of the $A - D - E$ group. From now on we denote these $\mathcal{N} = 2$ minimal model as $M(G, N)$ ($G = A_m$, D_m, E_m), or more simply as M_N when there is no problem of confusion. Hence, the model to be studied is the RNS superstring compactified on $\mathbf{R}^{d-1,1} \times (\mathbf{R}_\phi \times S^1) \times M_N$.

The condition of critical dimension can be written as

$$\frac{N-2}{N} + (1 + Q^2) = n(\equiv \frac{10-d}{2}), \tag{2.3}$$

and it is easy to evaluate Q for each of the cases $d = 6, 4, 2$

$$\begin{aligned} d = 6;\quad Q &= \sqrt{\frac{2}{N}}, \\ d = 4;\quad Q &= \sqrt{\frac{N+2}{N}}, \\ d = 2;\quad Q &= \sqrt{\frac{2(N+1)}{N}}. \end{aligned} \tag{2.4}$$

Notice that the criticality condition (2.3) is equivalent to the Calabi-Yau condition for the non-compact n-fold (defined in a suitable weighted projective space)

$$\tilde{F}(x, z^1, z^2, \ldots, z^{n+1}) \equiv -\hat{\mu} x^{-\frac{2}{Q^2}} + F(z^1, z^2, \ldots, z^{n+1}) = 0. \tag{2.5}$$

In refs. [2, 3] the negative power term in the superpotential $W \sim \hat{\mu} x^{-\frac{2}{Q^2}}$ is replaced by the Kazama-Suzuki model for $SL(2, \mathbf{R})/U(1)$ with the level $k' \equiv \dfrac{2}{Q^2} + 2$. The equivalence between such a non-compact Kazama-Suzuki model and the $\mathcal{N} = 2$ Liouville theory (2.1) ($\hat{\mu}$

[1]More rigorous setup may be the "double scaling limit" discussed in [5, 6];

$$g_s \ \to\ 0, \quad \mu\ \to\ 0 \quad \text{with } \frac{\mu^{Q^2/2}}{g_s} \text{ fixed to be a sufficiently large value,}$$

so that the theory is weakly coupled.

corresponds to the cosmological constant μ in (2.2)) was discussed in [5] and it was pointed out that both theories are related by a kind of T-duality. We argue for this equivalence from the point of view of the free field realization in the Appendix B.

3 Toroidal Partition Functions

We study the toroidal partition functions for the above non-critical superstring models. The toroidal partition function for RNS superstring has the following general structure;

$$Z = \int \frac{d^2\tau}{\tau_2^2} Z_0(\tau,\bar{\tau}) Z_{GSO}(\tau,\bar{\tau}), \tag{3.1}$$

where $\tau \equiv \tau_1 + i\tau_2$ is the modulus of the torus ($d^2\tau/\tau_2^2$ is the modular invariant measure). Z_{GSO} denotes the part of the partition function which consists of those contributions on which the GSO projection acts non-trivially. We write the remaining part as Z_0.

Obviously Z_0 includes only the contributions from the transverse non-compact bosonic coordinates $\mathbf{R}^{d-2} \times \mathbf{R}_\phi$. The Liouville sector $\mathbf{R}_\phi$ is slightly non-trivial because of the existence of the background charge. We should bear in our mind that only the normalizable states contribute to the partition function. The normalizable spectrum (in the sense of the delta function normalization because the spectrum is continuous) in Liouville theory has the lower bound $h = \frac{Q^2}{8}$ [29, 20]. Since this lower bound is non-zero, we must be careful in the integration over the zero-mode momentum. The result, however, turns out to be the same as that of the standard non-compact free boson without background charge,

$$\begin{aligned} Z_L(\tau,\bar{\tau}) &= \frac{1}{|\prod_{n=1}(1-q^n)|^2} \int_{-\frac{iQ}{2}-\infty}^{-\frac{iQ}{2}+\infty} dp \ \exp\left(-4\pi\tau_2\Big(\frac{1}{2}p^2 + \frac{i}{2}pQ - \frac{c_L}{24}\Big)\right), \\ &= \frac{1}{|\prod_{n=1}(1-q^n)|^2} \int_{-\infty}^{+\infty} dp \ \exp\left(-4\pi\tau_2\Big(\frac{1}{2}p^2 + \frac{1}{8}Q^2 - \frac{c_L}{24}\Big)\right) = \frac{1}{\tau_2^{1/2}|\eta(\tau)|^2} \end{aligned} \tag{3.2}$$

where $c_L = 1 + 3Q^2$. It is well-known [29] that the effective value of the Liouville central charge $c_{\text{eff},L}$ is equal to

$$c_{\text{eff},L} \equiv c_L - 24 \times \frac{Q^2}{8} = 1, \tag{3.3}$$

irrespective of the value of Q and the dependence on the background charge disappears from the net result.

In this way we obtain

$$Z_0(\tau,\bar{\tau}) = \frac{1}{\tau_2^{(d-1)/2}|\eta(\tau)|^{2(d-1)}}. \tag{3.4}$$

This expression is manifestly modular invariant.

The part Z_{GSO} is rather non-trivial. We need consider it separately in the cases $d = 6, 4, 2$. We first discuss the simplest case $d = 6$, and then proceed to the $d = 2, 4$ cases.

3.1 $d = 6$ Case

Let $\mathcal{H}^{(NS)}_{lm}$ ($\mathcal{H}^{(R)}_{lm}$) be the (left-moving) Hilbert space of the NS (R) sector of CFT describing the minimal model M_N. The spectra of $U(1)_R$-charges and conformal weights are given by;

$$\begin{array}{lll} \mathcal{H}^{(NS)}_{lm}\ (l+m \equiv 0\ (\mathrm{mod}\ 2)): & q = \dfrac{m}{N} + n\ (n \in \mathbf{Z}), & h = \dfrac{l(l+2)-m^2}{4N} + n\ (n \in \dfrac{1}{2}\mathbf{Z}_{\geq 0}), \\ \mathcal{H}^{(R)}_{lm}\ (l+m \equiv 1\ (\mathrm{mod}\ 2)): & q = \dfrac{m}{N} - \dfrac{1}{2} + n\ (n \in \mathbf{Z}), & h = \dfrac{l(l+2)-m^2}{4N} + \dfrac{1}{8} + n\ (n \in \mathbf{Z}_{\geq 0}). \end{array} \tag{3.5}$$

We also consider the (left-moving) Fock space $\mathcal{H}_p$ of the bosonic coordinate Y of S^1 constructed on the Fock vacuum $|p\rangle$, $\oint i\partial Y|p\rangle = p|p\rangle$. Values of the momenta p are chosen so that they are compatible with the GSO projection condition.

The total $\mathcal{N} = 2$ $U(1)_R$-charge is given by

$$\begin{aligned} J_0^{(NS)} &= F + (F_{M_N} + \frac{m}{N}) - pQ, \\ J_0^{(R)} &= F + (F_{M_N} + \frac{m}{N} - \frac{1}{2}) - pQ, \end{aligned} \tag{3.6}$$

where F denotes the fermion number of $\mathbf{R}^{d-2} \times (\mathbf{R}_\phi \times S^1)$ sector and F_{M_N} denotes the fermion number of the M_N sector.

After these preparations conditions for the GSO projection (the conditions for the mutual locality with the space-time SUSY charges) can now be written as

- NS-sector

$$F + F_{M_N} + \frac{m}{N} - pQ \in 2\mathbf{Z} + 1, \tag{3.7}$$

- R-sector

$$F + F_{M_N} + \frac{m}{N} - pQ \in 2\mathbf{Z}. \tag{3.8}$$

Let us now compute the trace over the left-moving Hilbert space. For instance, let us suppose $F+F_{M_N}=$ even, and consider the NS-sector. Then we have $p=\frac{1}{Q}\left(2n+1+\frac{m}{N}\right)$ ($n\in\mathbf{Z}$), and the sum over momenta becomes,

$$\sum_n q^{\frac{1}{2}p^2}=\sum_n q^{\frac{N}{4}(2n+1+\frac{m}{N})^2}=\sum_n q^{N(n+\frac{m+N}{2N})^2}=\Theta_{m+N,N}(\tau). \tag{3.9}$$

When combined with factors coming from oscillator modes and the minimal model M_N, NS sector partition function becomes

$$\frac{1}{2}\left\{\left(\frac{\theta_3}{\eta}\right)^3\mathrm{ch}^{(NS)}_{lm}+\left(\frac{\theta_4}{\eta}\right)^3\tilde{\mathrm{ch}}^{(NS)}_{lm}\right\}\frac{\Theta_{m+N,N}}{\eta}.$$

Here $\eta=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and $\theta_3=\sum_n q^{\frac{n^2}{2}}$, $\theta_4=\sum_n(-1)^n q^{\frac{n^2}{2}}$, $\theta_2=\sum_n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}$, $\Theta_{m,N}=\sum_n q^{N\left(n+\frac{m}{2N}\right)^2}$ ($q\equiv e^{2\pi i\tau}$) are the standard theta functions. $\mathrm{ch}^{(NS)}_{lm}(\tau)$, $\tilde{\mathrm{ch}}^{(NS)}_{lm}(\tau)$ denote the irreducible characters of $\mathcal{N}=2$ minimal model for NS-sector. (We summarize the definitions of these functions in the appendix A.) Similarly we can calculate the trace in other sectors, and obtain (we omit the factors of η-functions for simplicity),

$$\begin{aligned}\sum_{l=0}^{N-2}G_l \;=\; \frac{1}{2}\sum_{l=0}^{N-2}\sum_{m\in\mathbf{Z}_{2N}}&\Big\{\theta_3^3\mathrm{ch}^{(NS)}_{lm}(\Theta_{m,N}+\Theta_{m+N,N})-\theta_4^3\tilde{\mathrm{ch}}^{(NS)}_{lm}(\Theta_{m,N}-\Theta_{m+N,N})\\ &-\theta_2^3\mathrm{ch}^{(R)}_{lm}(\Theta_{m,N}+\Theta_{m+N,N})\Big\}.\end{aligned} \tag{3.10}$$

The above sum (3.10), however, counts each state twice due to the symmetry $G_l=G_{N-2-l}$. To avoid this double counting, we may define

$$F_l\stackrel{\mathrm{def}}{=}\frac{1}{2}\sum_{m\in\mathbf{Z}_{2N}}\Theta_{m,N}(\theta_3^3\mathrm{ch}^{(NS)}_{lm}-\theta_4^3\tilde{\mathrm{ch}}^{(NS)}_{lm}-\theta_2^3\mathrm{ch}^{(R)}_{lm}). \tag{3.11}$$

and have

$$G_l=F_l+F_{N-2-l}. \tag{3.12}$$

The desired partition sum then takes the form

$$Z_{GSO}(\tau,\bar{\tau})=\frac{1}{|\eta(\tau)|^8}\sum_{l,\bar{l}=0}^{N-2}N_{l,\bar{l}}F_l(\tau)F_{\bar{l}}(\bar{\tau}). \tag{3.13}$$

Thanks to the branching relation (A.12) we may rewrite F_l in the following simple form [3];

$$F_l(\tau)=\frac{1}{2}\left(\theta_3^4-\theta_4^4-\theta_2^4\right)\chi_l^{(N-2)}(\tau), \tag{3.14}$$

where $\chi_l^{(k)}(\tau)$ denotes the $\widehat{SU}(2)_k$ character of the spin $l/2$ representation. Hence the expression (3.13) is manifestly modular invariant when $N_{l,\bar{l}}$ is chosen to be one of the modular invariants $L_{l,\bar{l}}^{(N-2)}$ of $\widehat{SU}(2)_{N-2}$ theory which fulfill the conditions,

$$
\begin{aligned}
&L_{l,\bar{l}}^{(k)} = 0, \quad \text{unless } \frac{l(l+2)}{4(k+2)} - \frac{\bar{l}(\bar{l}+2)}{4(k+2)} \in \mathbf{Z}, \quad L_{k-l,k-\bar{l}}^{(k)} = L_{l,\bar{l}}^{(k)}, \\
&\sum_{l,\bar{l}} L_{l,\bar{l}}^{(k)} S_{ll'}^{(k)} S_{\bar{l}\bar{l}'}^{(k)} = L_{l',\bar{l}'}^{(k)}, \quad S_{ll'}^{(k)} \stackrel{\text{def}}{=} \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(l+1)(l'+1)}{k+2}\right).
\end{aligned}
\tag{3.15}
$$

Furthermore, F_l (3.14) identically vanishes by virtue of the Jacobi's abstruse identity: this is consistent with the existence of space-time SUSY.[2]

The general solutions $L_{l,\bar{l}}^{(N-2)}$ of (3.15) were completely classified by the $A-D-E$ series in ref. [22]. In these solutions the values of spin $l/2$ are in a one-to-one correspondence with the exponents of $A-D-E$ Lie algebra, and hence to each of the relevant deformations of the singularity $F=0$. In this way we obtain the modular invariants classified by the $A-D-E$ series corresponding to the singularity type of X_n [3, 21].

The appearance of the affine $SU(2)$ character in the expression (3.14) is quite expected. One may relate the background of degenerate K3 surface to a collection of NS5-branes by means of T-duality [3, 23], and it is well-known [24] that the world-sheet conformal field theory in the NS5 background contains the $SU(2)$ WZW model.

3.2 $d=2$ Case

In the case of $d=2$ the GSO conditions are given as follows;

- NS-sector

$$F + F_{M_N} + \frac{m}{N} - pQ \in 2\mathbf{Z}+1, \tag{3.16}$$

- R-sector

$$F + F_{M_N} + \frac{m}{N} - pQ \in 2\mathbf{Z}+1. \tag{3.17}$$

[2]As discussed in [19, 20], we only have the SUSY along the "boundary" $\mathbf{R}^{d-1,1}$, and no SUSY in the whole bulk space including the throat sector. Nevertheless we can conclude that the partition function should vanish in all the genera. See [20] for the detail.

We again determine the spectrum of the momenta p by imposing these conditions, and the trace for the left-movers is calculated as follows,

$$\sum_{l=0}^{N-2} G_l = \sum_{l=0}^{N-2} \frac{1}{2} \sum_{m\in \mathbf{Z}_{2N}} \Big\{\theta_3 \mathrm{ch}^{(NS)}_{lm}(\Theta_{\frac{m}{N+1},\frac{N}{N+1}} + \Theta_{\frac{m+N}{N+1},\frac{N}{N+1}}) - \theta_4 \tilde{\mathrm{ch}}^{(NS)}_{lm}(\Theta_{\frac{m}{N+1},\frac{N}{N+1}} - \Theta_{\frac{m+N}{N+1},\frac{N}{N+1}}) - \theta_2 \mathrm{ch}^{(R)}_{lm}(\Theta_{\frac{m}{N+1},\frac{N}{N+1}} + \Theta_{\frac{m+N}{N+1},\frac{N}{N+1}})\Big\}. \tag{3.18}$$

We again have $G_l = G_{N-2-l}$ and likewise introduce

$$F_l \stackrel{\text{def}}{=} \frac{1}{2} \sum_{m\in \mathbf{Z}_{2N}} \Big\{\Theta_{\frac{m}{N+1},\frac{N}{N+1}}(\theta_3 \mathrm{ch}^{(NS)}_{lm} - \theta_4 \tilde{\mathrm{ch}}^{(NS)}_{lm}) - \Theta_{\frac{m+N}{N+1},\frac{N}{N+1}} \theta_2 \mathrm{ch}^{(R)}_{lm}\Big\}, \tag{3.19}$$

$$G_l = F_l + F_{N-2-l}, \tag{3.20}$$

to avoid the double counting of states. However, it is easy to see that F_l here does not have a good modular transformation property. This is because the theta functions appearing in (3.19) have fractional levels, and they do not close among themselves under the modular transformation.

In order to avoid this difficulty we further decompose F_l into a set of functions $F_{l,r}$ ($r = 0, 1, \cdots, 2N+1$) with the help of the formula (A.5) as

$$F_l = \sum_{\substack{r\in \mathbf{Z}_{2(N+1)} \\ l+r\equiv 0 (\mathrm{mod}\ 2)}} F_{l,r}, \tag{3.21}$$

$$F_{l,r} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{m\in \mathbf{Z}_{2N}} \Theta_{(N+1)m+Nr, N(N+1)} \Big\{\theta_3 \mathrm{ch}^{(NS)}_{lm} - (-1)^{l+r}\theta_4 \tilde{\mathrm{ch}}^{(NS)}_{lm} - \theta_2 \mathrm{ch}^{(R)}_{lm}\Big\}. \tag{3.22}$$

$F_{l,r}$ has the symmetry

$$F_{l,r} = F_{l,r+2(N+1)} = F_{N-2-l,r+(N+1)}, \tag{3.23}$$

and thus we have

$$F_{N-2-l} = \sum_{\substack{r\in \mathbf{Z}_{2(N+1)} \\ l+r\equiv 1 (\mathrm{mod}\ 2)}} F_{l,r}. \tag{3.24}$$

It turns out that the functions $F_{l,r}(\tau)$ have the good modular properties as;

$$F_{l,r}(\tau+1) = e^{2\pi i \left\{\frac{l(l+2)}{4N} - \frac{N-2}{8N} - \frac{r^2}{4(N+1)} + \frac{3+(-1)^{l+r}}{8}\right\}} F_{l,r}(\tau), \tag{3.25}$$

$$F_{l,r}(-\frac{1}{\tau}) = (\sqrt{-i\tau})^2 \sum_{l'=0}^{N-2} \sum_{r'\in \mathbf{Z}_{2(N+1)}} \mathcal{S}_{(l,r)\,(l',r')}\, F_{l',r'}(\tau), \tag{3.26}$$

$$\mathcal{S}_{(l,r)\,(l',r')} \stackrel{\text{def}}{=} \frac{(-1)^{l+r} + (-1)^{l'+r'}}{2} S^{(N-2)}_{ll'} \frac{1}{\sqrt{N+1}} e^{2\pi i \frac{rr'}{2(N+1)}}. \tag{3.27}$$

Therefore, it seems reasonable to regard the functions $F_{l,r}(\tau)$ as the basic conformal blocks of our partition function. Note that due to (3.23) the two sets $\{F_{l,r};\ l+r \equiv 0 \pmod 2\}$ and $\{F_{l,r};\ l+r \equiv 1 \pmod 2\}$ are not independent and we should choose one of these as the building block of the theory. Let us take the set with $l+r \equiv 0 \pmod 2$. This restriction is consistent since (3.27) implies that the modular transformations act separately for each set.

Thanks to the transformation laws (3.25), (3.26) we can now construct the modular invariant partition function in the following form,

$$\begin{aligned} Z_{GSO}(\tau,\bar{\tau}) &= \frac{1}{|\eta(\tau)|^4} \sum_{l,\bar{l}=0}^{N-2} \sum_{r,\bar{r}\in \mathbf{Z}_{2(N+1)}} N_{(l,r),(\bar{l},\bar{r})} F_{l,r}(\tau) F_{\bar{l},\bar{r}}(\bar{\tau}), \\ N_{(l,r),(\bar{l},\bar{r})} &= L^{(N-2)}_{l,\bar{l}} M^{(N+1)}_{r,\bar{r}}. \end{aligned} \tag{3.28}$$

Here $L^{(N-2)}_{l,\bar{l}}$ again denotes one of the $A-D-E$ modular invariants of $\widehat{SU}(2)_{N-2}$, and $M^{(k)}_{r,\bar{r}}$ is the modular invariant of the "theta system" which satisfies the following conditions,

$$\begin{aligned} & M^{(k)}_{r,\bar{r}} = 0, \quad \text{unless } \frac{r^2}{4k} - \frac{\bar{r}^2}{4k} \in \mathbf{Z}, \quad M^{(k)}_{r+k,\,\bar{r}+k} = M^{(k)}_{r,\bar{r}} \\ & \sum_{r,\bar{r}} M^{(k)}_{r,\bar{r}} R^{(k)\dagger}_{r,r'} R^{(k)\dagger}_{\bar{r},\bar{r}'} = M^{(k)}_{r',\bar{r}'}, \quad R^{(k)}_{r,r'} \overset{\text{def}}{=} \frac{1}{\sqrt{2k}} e^{-2\pi i \frac{rr'}{2k}}. \end{aligned} \tag{3.29}$$

The simplest solution for $M^{(k)}_{r,\bar{r}}$ is, of course, given by $M^{(k)}_{r,\bar{r}} = \delta_{r,\bar{r}}$ (or $M^{(N-2)}_{r\bar{r}} = \delta_{r,-\bar{r}}$), and the most general solution is given by [25]

$$M^{(k)}_{r,\bar{r}} = \frac{1}{2} \sum_{x\in \mathbf{Z}_{2\beta},\, y\in \mathbf{Z}_{2\alpha}} \delta_{r,\alpha x+\beta y} \delta_{\bar{r},\alpha x-\beta y}, \tag{3.30}$$

where α, β are general integers such that $\alpha\beta = k$.

A few comments are in order:

1. Our solution (3.28) for the simplest case $N=2$ (the minimal model M_N becomes trivial) coincides with the one presented in ref.[20].

2. It is possible to derive the following relations for the functions $F_{l,r}$ by making use of the product formula of theta functions (A.4);

$$\begin{aligned} \sum_{\substack{r\in \mathbf{Z}_{2(N+1)} \\ l+r\equiv 0 \pmod 2}} \Theta_{r,N+1}(\tau,0) F_{l,r}(\tau,z) &= \frac{1}{2}\chi^{(N-2)}_l(\tau,0) \\ & \times \left\{ (\theta_3^2-\theta_4^2)(\tau,z)\Theta_{0,1}(\tau,2z) - (\theta_2^2+\theta_1^2)(\tau,z)\Theta_{1,1}(\tau,2z) \right\}, \end{aligned} \tag{3.31}$$

$$\sum_{\substack{r\in\mathbf{Z}_{2(N+1)}\\ l+r\equiv 1 \pmod 2}} \Theta_{r,N+1}(\tau,0)F_{l,r}(\tau,z) = \frac{1}{2}\chi_l^{(N-2)}(\tau,0)$$

$$\times\left\{(\theta_3^2+\theta_4^2)(\tau,z)\Theta_{1,1}(\tau,2z) - (\theta_2^2-\theta_1^2)(\tau,z)\Theta_{0,1}(\tau,2z)\right\}, \quad (3.32)$$

$$F_{l,r}(\tau,z) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{m\in\mathbf{Z}_{2N}} \Theta_{(N+1)m+Nr,N(N+1)}(\tau,-\frac{2z}{N})$$

$$\times\left(\theta_3 \mathrm{ch}_{lm}^{(NS)} - (-1)^{l+r}\theta_4\tilde{\mathrm{ch}}_{lm}^{(NS)} - \theta_2\mathrm{ch}_{lm}^{(R)} - i(-1)^{l+r}\theta_1\tilde{\mathrm{ch}}_{lm}^{(R)}\right)(\tau,z). \quad (3.33)$$

(Note $F_{l,r}(\tau) \equiv F_{l,r}(\tau, z=0)$). It is known [30] that the combination of theta functions in the right-hand-side of (3.31),(3.32) vanishes identically

$$(\theta_3^2-\theta_4^2)(\tau,z)\Theta_{0,1}(\tau,2z) - (\theta_2^2+\theta_1^2)(\tau,z)\Theta_{1,1}(\tau,2z) = 0, \quad (3.34)$$

$$(\theta_3^2+\theta_4^2)(\tau,z)\Theta_{1,1}(\tau,2z) - (\theta_2^2-\theta_1^2)(\tau,z)\Theta_{0,1}(\tau,2z) = 0. \quad (3.35)$$

Thus the sum of functions $F_{l,r}$ (3.31), (3.32) in fact vanishes identically. Then these equations imply that the functions $F_{l,r}$ themselves should vanish separately for each $|r|$ since the level-$(N+1)$ theta functions $\Theta_{r,N+1}(\tau,0)$ are functionally independent for different $|r|$. We have explicitly verified by Maple that $F_{l,r}(\tau,z) + F_{l,-r}(\tau,z)$ in fact vanishes in lower orders in $q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi iz}$ and y^{-1}, for every l, r and $N = 2,3,4$.

We conjecture that the identity

$$F_{l,r}(\tau,z) + F_{l,-r}(\tau,z) \equiv 0 \quad (3.36)$$

holds for arbitrary l, r, N. If this is the case, we have $F_{l,r}(\tau) \equiv F_{l,r}(\tau,0) \equiv F_{l,-r}(\tau,0) = 0$ and the partition function vanishes $Z_{GSO} = 0$: this is consistent with the presence of the space-time supersymmetry.

3. The modular properties of $F_{l,r}(\tau)$ (3.25), (3.26) can be immediately read off from (3.31), (3.32): we find that the index l of $F_{l,r}$ transforms like the spin of the representation of affine $SU(2)$ and the index r transforms like a label of the $U(1)$ theta function. Modular properties of $F_{l,r}(\tau)$ is in fact similar to those of parafermionic theory [25].

4. $F_{l,r}(\tau,z)$ is transformed under the spectral flow $z \mapsto z + \frac{\alpha}{2}\tau$ ($\alpha \in \mathbf{Z}$) [15, 31] as follows;

$$F_{l,r}(\tau, z+\frac{\alpha}{2}\tau) = (-1)^{\alpha} q^{-\frac{\alpha^2}{2}} y^{-2\alpha} F_{l,r}(\tau,z). \quad (3.37)$$

This means that $F_{l,r}(\tau,z)$ is a "flow-invariant orbit" in the sense of [31]. This fact justifies regarding $F_{l,r}$ as the building block of the partition function.

3.3 $d=4$ Case

The GSO conditions are given as follows;

- NS-sector

$$F+F_{M_N}+\frac{m}{N}-pQ\in 2\mathbf{Z}+1, \tag{3.38}$$

- R-sector

$$F+F_{M_N}+\frac{m}{N}-pQ\in 2\mathbf{Z}+\frac{1}{2}. \tag{3.39}$$

In this case the trace over the left-moving Hilbert space is given by,

$$\begin{aligned}\sum_{l=0}^{N-2}G_l \;=\; &\sum_{l=0}^{N-2}\frac{1}{2}\sum_{m\in\mathbf{Z}_{2N}}\left\{\theta_3^2\mathrm{ch}_{lm}^{(NS)}(\Theta_{\frac{2m}{N+2},\frac{2N}{N+2}}+\Theta_{\frac{2(m+N)}{N+2},\frac{2N}{N+2}})-\theta_4^2\tilde{\mathrm{ch}}_{lm}^{(NS)}(\Theta_{\frac{2m}{N+2},\frac{2N}{N+2}}-\Theta_{\frac{2(m+N)}{N+2},\frac{2N}{N+2}})\right.\\ &\left.-\theta_2^2\mathrm{ch}_{lm}^{(R)}(\Theta_{\frac{2m+N}{N+2},\frac{2N}{N+2}}+\Theta_{\frac{2m-N}{N+2},\frac{2N}{N+2}})\right\}.\end{aligned} \tag{3.40}$$

Again G_l consists of theta functions with fractional levels and we have $G_l=G_{N-2-l}$. We introduce $F_l=\frac{1}{2}G_l$ and expand F_l into a set of functions $F_{l,r}$ $(r=0,1,\cdots,2N+3)$,

$$\begin{aligned}F_{l,r} &\stackrel{\mathrm{def}}{=} \frac{1}{4}\sum_{m\in\mathbf{Z}_{4N}}\Theta_{(N+2)m+Nr,2N(N+2)}\left\{\theta_3^2\mathrm{ch}_{lm}^{(NS)}-(-1)^{\frac{r+m}{2}}\theta_4^2\tilde{\mathrm{ch}}_{lm}^{(NS)}-\theta_2^2\mathrm{ch}_{lm}^{(R)}\right\} \quad (3.41)\\ &\qquad (l+r\equiv 0 \pmod 2)),\\ F_{l,r} &\stackrel{\mathrm{def}}{=} 0, \qquad (l+r\equiv 1 \pmod 2)),\\ F_l &= \sum_{r\in\mathbf{Z}_{2(N+2)}}F_{l,r}. \quad (3.42)\end{aligned}$$

It is easy to see that

$$F_{l,r}=F_{l,\,r+2(N+2)}=F_{N-2-l,\,r+(N+2)}. \tag{3.43}$$

Note that (3.43) is consistent with the definition $F_{l,r}\equiv 0$ $(l+r\equiv 1 \pmod 2))$, since $l+r\equiv l+r+2(N+2)\equiv (N-2-l)+(r+N+2) \pmod 2$.

$F_{l,r}$ possess the following modular transformation properties;

$$\begin{aligned}F_{l,r}(\tau+1) &= e^{2\pi i\left\{\frac{l(l+2)}{4N}-\frac{N-2}{8N}-\frac{r^2}{4(N+2)}+\frac{1}{2}\right\}}F_{l,r}(\tau), \quad (3.44)\\ F_{l,r}(-\frac{1}{\tau}) &= (\sqrt{-i\tau})^3\sum_{l'=0}^{N-2}\sum_{r'\in\mathbf{Z}_{2(N+2)}}\mathcal{S}_{(l,r)\,(l',r')}F_{l',r'}(\tau), \quad (3.45)\\ \mathcal{S}_{(l,r)\,(l',r')} &\stackrel{\mathrm{def}}{=} S_{ll'}^{(N-2)}\frac{1}{\sqrt{2(N+2)}}e^{2\pi i\frac{rr'}{2(N+2)}}, \quad (3.46)\end{aligned}$$

It is now easy to construct modular invariant partition functions

$$\begin{aligned} Z_{GSO}(\tau,\bar\tau) &= \frac{1}{|\eta(\tau)|^6}\sum_{l,\bar l=0}^{N-2}\sum_{r,\bar r\in\mathbf{Z}_{2(N+2)}} N_{(l,r),(\bar l,\bar r)}F_{l,r}(\tau)F_{\bar l,\bar r}(\bar\tau), \\ N_{(l,r),(\bar l,\bar r)} &= \frac{1}{2}\left(L^{(N-2)}_{l,\bar l}M^{(N+2)}_{r,\bar r} + L^{(N-2)}_{N-2-l,\bar l}M^{(N+2)}_{r+N+2,\bar r}\right), \end{aligned} \tag{3.47}$$

where $L^{(k)}_{l,\bar l}$ and $M^{(k)}_{r,\bar r}$ are defined as before.

The solution for the simplest case $N=2$ (the case of conifold singularity in CY_3 [2]) was first obtained by S. Mizoguchi from a somewhat different approach [32].

As in the two-dimensional case, we can construct the following combination of the $F_{l,r}$ functions

$$\sum_{r\in\mathbf{Z}_{2(N+2)}}\Theta_{r,N+2}(\tau,0)F_{l,r}(\tau,z) = \frac{1}{4}\chi^{(N-2)}_l(\tau,0)\left(\theta_3^4(\tau,z)-\theta_4^4(\tau,z)-\theta_2^4(\tau,z)+\theta_1^4(\tau,z)\right), \tag{3.48}$$

$$\begin{aligned} F_{l,r}(\tau,z) &\stackrel{\text{def}}{=} \frac{1}{4}\sum_{m\in\mathbf{Z}_{4N}}\Theta_{(N+2)m+Nr,2N(N+2)}(\tau,-\frac{z}{N}) \\ &\times\left(\theta_3^2\mathrm{ch}^{(NS)}_{lm} - (-1)^{\frac{r+m}{2}}\theta_4^2\tilde{\mathrm{ch}}^{(NS)}_{lm} - \theta_2^2\mathrm{ch}^{(R)}_{lm} + i(-1)^{\frac{r+m}{2}}\theta_1^2\tilde{\mathrm{ch}}^{(R)}_{lm}\right)(\tau,z), \quad (l+r\equiv 0 \ (\mathrm{mod}\ 2)) \\ F_{l,r}(\tau,z) &\stackrel{\text{def}}{=} 0, \qquad (l+r\equiv 1\ (\mathrm{mod}\ 2)). \end{aligned} \tag{3.49}$$

We note that the right-hand-side of (3.48) vanishes due to Jacobi's identity. Then as in the case of two-dimensional theories, we expect that functions $F_{l,r}$ should vanish separately for each $|r|$, $F_{l,r}(\tau,z)+F_{l,-r}(\tau,z)\equiv 0$. We have explicitly checked this for lower orders of q, y, y^{-1} by Maple and found that in fact a stronger relation

$$F_{l,r}(\tau,z)\equiv 0, \tag{3.50}$$

holds. We conjecture that (3.50) holds for all l,r,N. In this case all the modular invariant theories again have vanishing partition functions and are consistent with the presence of space-time supersymmetry.

We may again read off the modular transformation rule (3.46) from the identity (3.48): $F_{l,r}$ transforms like an affine $\widehat{SU}(2)$ character in its index l and like $U(1)$ theta function in its label r.

We may show

$$F_{l,r}(\tau,z+\frac{\alpha}{2}\tau) = (-1)^\alpha q^{-\frac{\alpha^2}{2}}y^{-2\alpha}F_{l,r}(\tau,z), \tag{3.51}$$

which implies that the functions $F_{l,r}(\tau,z)$ are the flow-invariant orbits for each l,r.

4 Conclusions

In this paper we have constructed the toroidal partition functions of the non-critical superstring theory on $\mathbf{R}^{d-1,1} \times (\mathbf{R}_\phi \times S^1_Y) \times M_N$, which is to provide the dual description of the singular Calabi-Yau compactification in the decoupling limit. We have found that there exists a natural $A-D-E$ classification of modular invariants associated to the type of Calabi-Yau singularities in all cases of $d = 6, 4, 2$. In cases $d = 4, 2$, we found that the conformal blocks composing the partition function behave like primary fields of the parafermionic theory. It will be very interesting if we could identify our conformal blocks with suitable scaling operators in respective field theories and elucidate their dynamical properties.

As we have discussed at the beginning of Section 3, the presence of the background charge in the Liouville sector creates a gap $h \geq Q^2/8$ in the CFT spectrum. In particular the graviton (which corresponds to $h = 0$) does not appear in the modular invariant partition function. Thus the system in fact describes some non-gravitational theory and the theory is interpreted as being at the decoupling limit of type II superstring. It is quite reassuring to us that one can construct modular invariant amplitudes for string propagation even in such a "singular" situation where some of the conventional world-sheet technology may break down and non-perturbative effects play an important role.

Landau-Ginzburg theory has the disturbing feature of the appearance of a negative power piece in the superpotential when applied to describe singular (non-compact) Calabi-Yau manifolds. It is not clear how to treat the negative power operator within the framework of the standard $\mathcal{N} = 2$ SCFT. It now appears, however, the negative power term may be handled properly by means of the Liouville degrees of freedom with an appropriate background charge. The appearance of the gap and the continuous spectrum above the gap in string propagation in singular Calabi-Yau manifold are reproduced exactly by the dynamics of the Liouville field. It will be extremely useful if we have a better understanding on the relationship between the singular geometry and the dynamics of Liouville field.

It will be interesting to consider more general class of $\mathcal{N} = 2$ models instead of $\mathcal{N} = 2$ minimal model (Landau-Ginzburg orbifolds [34] or Gepner models [33], etc.). Quite recently, in ref. [6], Landau-Ginzburg orbifolds are discussed, relating it to the $\mathcal{N} = 2$ $SCFT_4$ *with matter fields* [27]. It may also be interesting to study non-rational Calabi-Yau singularities (collapse

of del Pezzo surfaces etc.). These problems may be regarded as natural generalizations of the Gepner model, namely, the (orbifoldized) tensor product of minimal models ("compact models") with the Liouville theory ("non-compact" models). Construction of modular invariants for such models will provide important consistency checks of their dynamics.

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A Notations and Conventions

In this appendix we summarize our conventions and present some formulas used in the manuscript.

1. theta functions We set $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$ and introduce various theta functions;

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\exp(\frac{\pi i\tau}{4})\sin(\pi z)\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m),\\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\exp(\frac{\pi i\tau}{4})\cos(\pi z)\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m),\\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}),\\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}).
\end{aligned} \tag{A.1}$$

$$\Theta_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}. \tag{A.2}$$

We use the abbreviations; $\theta_i \equiv \theta_i(\tau,0)$ $(\theta_1\equiv 0)$, $\Theta_{m,k}(\tau)\equiv\Theta_{m,k}(\tau,0)$. We also define

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{A.3}$$

The product formula of theta function is written as [35, 36];

$$\Theta_{m,k}(\tau,z)\Theta_{m',k'}(\tau,z') = \sum_{r\in\mathbf{Z}_{k+k'}}\Theta_{mk'-m'k+2kk'r,kk'(k+k')}(\tau,u)\Theta_{m+m'+2kr,k+k'}(\tau,v), \tag{A.4}$$

where we set $u=\dfrac{z-z'}{k+k'}$, $v=\dfrac{kz+k'z'}{k+k'}$.

The following identity is often used (p is an integer);

$$\Theta_{m/p,k/p}(\tau,z) = \Theta_{m,k}(\tau/p,z/p) = \sum_{r\in\mathbf{Z}_p}\Theta_{m+2kr,pk}(\tau,z/p). \tag{A.5}$$

2. characters of $\mathcal{N}=2$ minimal model Let $\chi_l^{(k)}(\tau,z)$ be the character of $\widehat{SU}(2)_k$ with the spin $l/2$ $(0\le l\le k)$ representation;

$$\chi_l^{(k)}(\tau,z) = \frac{\Theta_{l+1,k+2}-\Theta_{-l-1,k+2}}{\Theta_{1,2}-\Theta_{-1,2}}(\tau,z)\ . \tag{A.6}$$

String function $c^l_m(\tau)$ is defined by

$$\chi^{(k)}_l(\tau,z) = \sum_{m\in\mathbf{Z}_{2k}} c^l_m(\tau)\Theta_{m,k}(\tau,z). \tag{A.7}$$

We introduce

$$\chi^{l,s}_m(\tau,z) = \sum_{r\in\mathbf{Z}_k} c^l_{m-s+4r}(\tau)\Theta_{2m+(k+2)(-s+4r),2k(k+2)}(\tau,z/(k+2)). \tag{A.8}$$

String function c^l_m has the following properties;

$$c^l_m = c^l_{-m} = c^l_{m+2k} = c^{k-l}_{m+k}, \quad c^l_m = 0 \text{ unless } l+m \equiv 0 \text{ (mod 2)}. \tag{A.9}$$

Likewise, $\chi^{l,s}_m$ has the properties;

$$\chi^{l,s}_m = \chi^{l,s}_{m+2(k+2)} = \chi^{l\,s+4}_m = \chi^{k-l,\,s+2}_{m+(k+2)}, \quad \chi^{l,s}_m = 0 \text{ unless } l+m+s \equiv 0 \text{ (mod 2)}, \tag{A.10}$$

and thus m, s run over the range $m \in \mathbf{Z}_{2(k+2)}$, $s \in \mathbf{Z}_4$.

The characters of $\mathcal{N}=2$ minimal model with $\hat{c} = \dfrac{k}{k+2}$ are defined [37, 33] by

$$\begin{aligned}
\mathrm{ch}^{(NS)}_{l,m}(\tau,z) &\equiv \mathrm{Tr}_{\mathcal{H}^{NS}_{l,m}} q^{L_0-\hat{c}/8} y^{J_0} = \chi^{l,0}_m + \chi^{l,2}_m \\
\tilde{\mathrm{ch}}^{(NS)}_{l,m}(\tau,z) &\equiv \mathrm{Tr}_{\mathcal{H}^{NS}_{l,m}} (-1)^F q^{L_0-\hat{c}/8} y^{J_0} = \chi^{l,0}_m - \chi^{l,2}_m \\
\mathrm{ch}^{(R)}_{l,m}(\tau,z) &\equiv \mathrm{Tr}_{\mathcal{H}^{R}_{l,m}} q^{L_0-\hat{c}/8} y^{J_0} = \chi^{l,1}_m + \chi^{l,3}_m \\
\tilde{\mathrm{ch}}^{(R)}_{l,m}(\tau,z) &\equiv \mathrm{Tr}_{\mathcal{H}^{R}_{l,m}} (-1)^F q^{L_0-\hat{c}/8} y^{J_0} = \chi^{l,1}_m - \chi^{l,3}_m.
\end{aligned} \tag{A.11}$$

It is easy to prove the following branching relation by means of the product formula of theta functions (A.4) [33, 36];

$$\chi^{(k)}_l(\tau,w)\Theta_{s,2}(\tau,w-z) = \sum_{m\in\mathbf{Z}_{2(k+2)}} \chi^{l,s}_m(\tau,z)\Theta_{m,k+2}(\tau,w-2z/(k+2)). \tag{A.12}$$

This relation (A.12) represents the minimal model as the Kazama-Suzuki coset for $SU(2)_k/U(1)$.

B Equivalence between the Kazama-Suzuki Model for $SL(2;\mathbf{R})/U(1)$ and the $\mathcal{N}=2$ Liouville Theory

In this appendix we discuss the equivalence between the $\mathcal{N}=2$ coset SCFT for $SL(2;\mathbf{R})/U(1)$ and the $\mathcal{N}=2$ Liouville theory from the viewpoint of free field realizations. We start from the following free field realization[3] of $SL(2;\mathbf{R})$-current algebra with the level $k+2$,

$$\begin{cases} J^3 = \sqrt{\dfrac{k+2}{2}}\partial u \\ J^{\pm} = -\left(\sqrt{\dfrac{k+2}{2}}i\partial X \mp \sqrt{\dfrac{k}{2}}\partial\phi\right) e^{\mp\sqrt{\frac{2}{k+2}}(u+iX)}, \end{cases} \tag{B.1}$$

where $X(z)X(0) \sim -\ln z$, $\phi(z)\phi(0) \sim -\ln z$, $u(z)u(0) \sim -\ln z$ are free scalar fields. The Sugawara stress tensor is given by

$$T_{SL(2;\mathbf{R})} = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k}}\partial^2\phi - \frac{1}{2}(\partial u)^2, \tag{B.2}$$

which possesses the central charge $c = 3 + \dfrac{6}{k}$. (k is related to N in the text as $k=N,\ d=6; k=2N/(N+2),\ d=4; k=N/(N+1),\ d=2$, respectively).

The Kazama-Suzuki model [17] for $SL(2;\mathbf{R})/U(1)$ is given by further tensoring the system with two $U(1)$-charged fermions; $\psi^+(z)\psi^-(0) \sim \dfrac{1}{z}$, and then by gauging the $U(1)$-subgroup. Here we adopt the BRST formulation and first bosonize the $U(1)$-gauge field as $A(z) \sim i\partial v(z)$, where $v(z)$ is a real scalar field with $v(z)v(0) \sim -\ln z$. We also bosonize the fermions $\psi^{\pm}$ in the standard fashion;

$$\psi^{\pm}(z) = e^{\pm iH(z)}, \quad H(z)H(0) \sim -\ln z. \tag{B.3}$$

The total stress tensor for the Kazama-Suzuki model then reduces to the following form,

$$T = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k}}\partial^2\phi - \frac{1}{2}(\partial u)^2 - \frac{1}{2}(\partial v)^2 - \frac{1}{2}(\partial H)^2 - \eta\partial\xi, \tag{B.4}$$

where (ξ,η) is the spin (0,1) ghost system and the BRST charge is given by

$$Q_{U(1)} = \oint \xi\left(\sqrt{\frac{k+2}{2}}\partial u + i\sqrt{\frac{k}{2}}\partial v + i\partial H\right). \tag{B.5}$$

[3] We have another familiar free field realization; "Wakimoto representation" [38] (φ, β, γ). The relation between these free fields and X, ϕ, u here is given as follows;

$$\begin{cases} \varphi = \phi + \sqrt{\dfrac{k}{k+2}}(u+iX), \\ \beta = -\left(\sqrt{\dfrac{k+2}{2}}i\partial X + \sqrt{\dfrac{k}{2}}\partial\phi\right) e^{\sqrt{\frac{2}{k+2}}(u+iX)}, \\ \gamma = e^{-\sqrt{\frac{2}{k+2}}(u+iX)}. \end{cases}$$

This stress tensor (B.4) has the correct central charge $c = 3(1+\frac{2}{k})$ and the world-sheet $\mathcal{N}=2$ superconformal symmetry is generated by the currents,

$$\begin{cases} G^{\pm} = \dfrac{1}{\sqrt{k}}\psi^{\pm}J^{\mp} = -\dfrac{1}{\sqrt{k}}\left(\sqrt{\dfrac{k+2}{2}}i\partial X \pm \sqrt{\dfrac{k}{2}}\partial\phi\right) e^{\pm\sqrt{\frac{2}{k+2}}(u+iX)\pm iH}, \\ J = \psi^{+}\psi^{-} + \dfrac{2}{k}(J^3+\psi^{+}\psi^{-}) = \dfrac{k+2}{k}i\partial H + \dfrac{\sqrt{2(k+2)}}{k}\partial u. \end{cases} \tag{B.6}$$

Now, let us try to reduce the above Kazama-Suzuki model to the $\mathcal{N}=2$ Liouville theory. For this purpose it is convenient to introduce the following field redefinition,

$$\begin{pmatrix} v' \\ H' \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \sqrt{\frac{k}{k+2}} & \sqrt{\frac{2}{k+2}} \\ -\sqrt{\frac{2}{k+2}} & \sqrt{\frac{k}{k+2}} \end{pmatrix} \begin{pmatrix} v \\ H \end{pmatrix}. \tag{B.7}$$

Clearly we have $v'(z)v'(0) \sim -\ln z$, $H'(z)H'(0) \sim -\ln z$ and $v'(z)H'(0) \sim 0$. The BRST-charge (B.5) is rewritten as

$$Q_{U(1)} = \sqrt{\frac{k+2}{2}}\oint \xi(\partial u + i\partial v'), \tag{B.8}$$

and the stress tensor (B.4) is reexpressed as follows,

$$T = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k}}\partial^2\phi - \frac{1}{2}(\partial H')^2 + \left\{Q_{U(1)},\ \frac{1}{\sqrt{2(k+2)}}\eta(-\partial u + i\partial v')\right\}. \tag{B.9}$$

We also obtain the following expressions for $G^{\pm}$, J,

$$\begin{cases} G^{\pm} = -\dfrac{1}{\sqrt{k}}\left(\sqrt{\dfrac{k+2}{2}}i\partial X \pm \sqrt{\dfrac{k}{2}}\partial\phi\right) e^{\pm i\left(\sqrt{\frac{k}{k+2}}H' + \sqrt{\frac{2}{k+2}}X\right) \pm\sqrt{\frac{2}{k+2}}(u+iv')}, \\ J = \sqrt{\dfrac{k+2}{k}}i\partial H' + \left\{Q_{U(1)},\ \dfrac{2}{k}\eta\right\}. \end{cases} \tag{B.10}$$

In order to eliminate u, v' in these formulas we set $U \stackrel{\text{def}}{=} e^{-\sqrt{\frac{2}{k+2}}\oint(u+iv')J}$ and perform the similarity transformation,

$$G'^{\pm} \stackrel{\text{def}}{=} UG^{\pm}U^{-1} = -\frac{1}{\sqrt{k}}\left(\sqrt{\frac{k+2}{2}}i\partial X \pm \sqrt{\frac{k}{2}}\partial\phi\right) e^{\pm i\left(\sqrt{\frac{k}{k+2}}H' + \sqrt{\frac{2}{k+2}}X\right)}. \tag{B.11}$$

Stress tensor and $U(1)$ current remain invariant $T' \stackrel{\text{def}}{=} UTU^{-1} = T$, $J' \stackrel{\text{def}}{=} UJU^{-1} = J$ up to $Q_{U(1)}$-exact terms due to following relations,

$$\begin{aligned} \left[-\sqrt{\frac{2}{k+2}}\oint(u+iv')J,\ J(z)\right] &= \left\{Q_{U(1)},\ -\frac{2}{k}\eta(z)\right\}, \\ \left[-\sqrt{\frac{2}{k+2}}\oint(u+iv')J,\ T(z)\right] &= 0 \end{aligned} \tag{B.12}$$

It is also obvious that $UQ_{U(1)}U^{-1} = Q_{U(1)}$, and hence this similarity transformation is in fact well-defined on the Hilbert space of Kazama-Suzuki model. Furthermore it is convenient to rotate again X, H' as,

$$\begin{pmatrix} Y \\ H'' \end{pmatrix} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \sqrt{\frac{k}{k+2}} & -\sqrt{\frac{2}{k+2}} \\ \sqrt{\frac{2}{k+2}} & \sqrt{\frac{k}{k+2}} \end{pmatrix} \begin{pmatrix} X \\ H' \end{pmatrix}, \tag{B.13}$$

and set $\Psi^{\pm} \stackrel{\mathrm{def}}{=} e^{\pm iH''}$. Then we finally obtain (up to $Q_{U(1)}$-exact terms)

$$\begin{cases} T' = -\dfrac{1}{2}(\partial Y)^2 - \dfrac{1}{2}(\partial\phi)^2 - \dfrac{1}{\sqrt{2k}}\partial^2\phi - \dfrac{1}{2}(\Psi^+\partial\Psi^- - \partial\Psi^+\Psi^-) \\ G'^{\pm} = -\dfrac{1}{\sqrt{2}}\Psi^{\pm}(i\partial Y \pm \partial\phi) \mp \dfrac{1}{\sqrt{k}}\partial\Psi^{\pm} \\ J' = \Psi^+\Psi^- - \sqrt{\dfrac{2}{k}}i\partial Y. \end{cases} \tag{B.14}$$

This is no other than the superconformal currents of the $\mathcal{N} = 2$ Liouville theory $\mathbf{R}_{\phi} \times S^1$ with the background charge $Q = \sqrt{\frac{2}{k}}$. Notice also that the Liouville potential $\delta S_{\pm} = \mu \int d^2z\, \Psi^{\pm}\bar{\Psi}^{\pm} e^{-\frac{1}{Q}(\phi \mp iY)}$ is actually a screening operator (it commutes with all of the superconformal currents (B.14)) and moreover $U(\delta S_{\pm})U^{-1} = \delta S_{\pm}$ holds. Thus we do not have to make modification in the above derivation of equivalence, even in the presence of such an interaction term.

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